# On the generalization of learning algorithms that do not converge

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# Non-converging optimization

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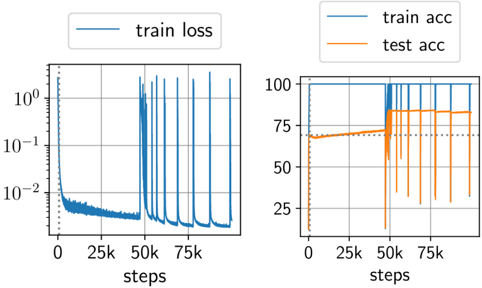


Courtesy: [Lyu Li Arora 2022]. Recent interest [Kong and Tao 2021, Cohen et al 2021, Lobacheva et al 2021, Zhang Li Sra Jadbabaie 2022] in non-converging training algorithms

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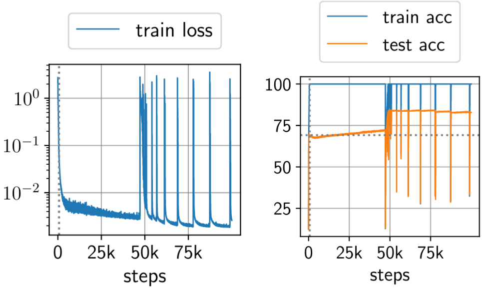


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(Q1) How can we *define* and *study* the generalization properties of a non-converging learning algorithm?

(Q2) Can the statistical/ergodic properties of the algorithm *predict* its generalization performance?

# SGD/GD dynamics on weight space:

$$w_{t+1} = w_t - \eta \hat{\nabla} L_S(w_t),$$

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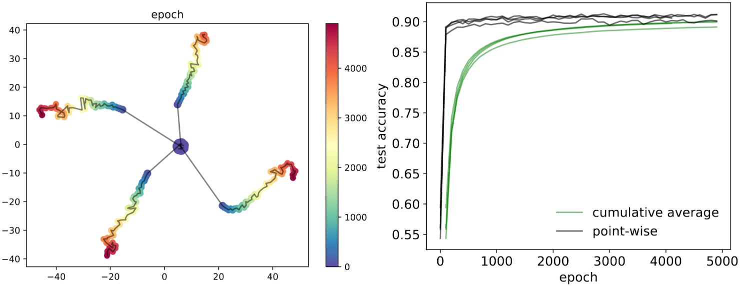
In general, deterministic/stochastic nonlinear dynamics on compact set. No guarantee of convergence to fixed points. There exist multiple invariant, ergodic distributions on weight space, $M$.

# Convergence of loss time-averages

**Assumption 1:** For almost every $w_0$ and every $z$, time-average of $\ell(z, \cdot)$ converges to a constant $\langle \ell_z \rangle_S$, independent of $w_0$.

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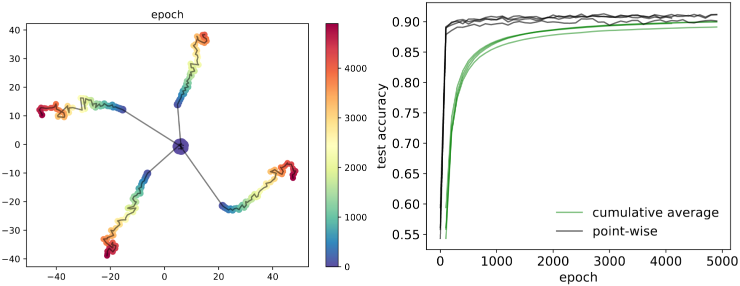


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Assumption allows us to extend algorithmic stability to *statistical algorithmic stability* (SAS).

# Statistical Algorithmic Stability

**Classical algorithmic stability [Bousquet and Elisseeff 2002]:**

$$\beta := \sup_{z} \sup_{S,S'} |\ell(z, w_S^*) - \ell(z, w_{S'}^*)|.$$

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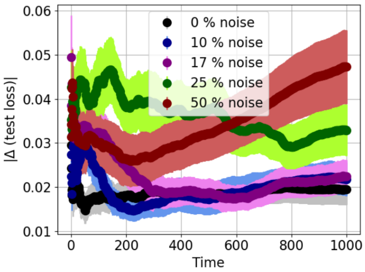
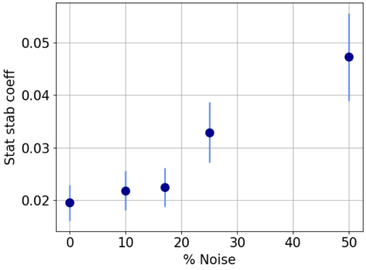
- applies to non-converging learning algorithms
- is constant on network function/parameter space

# Numerical approximation of β for SGD on VGG16 model trained on CIFAR10




Noisy CIFAR10 labels.
Anticlockwise: Sample mean over 45 $(S, S')$ pairs, with error bars, of time-averaged test loss difference. Lower bound on β with error bars computed as sample mean. Test loss vs. time (epoch).

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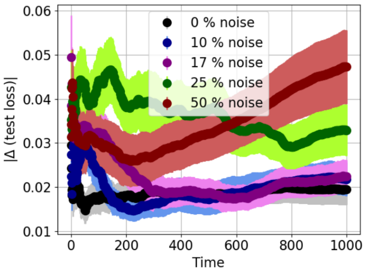


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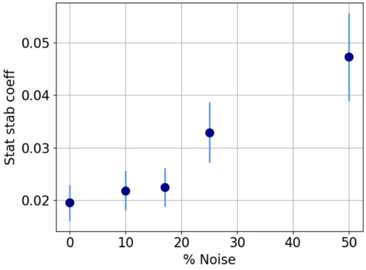


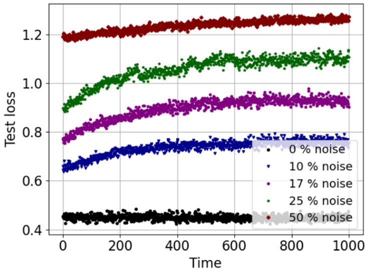

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Theorem 2 (**Slower convergence of loss statistics implies larger** $\beta$) Let $\lambda$ be the slowest mixing rate of the transition operators on loss space. Then, the corresponding training algorithm with $n$ samples has SAS coefficient

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